

PARAMETRIZING AN INTEGER LINEAR PROGRAM BY AN INTEGER

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ABSTRACT. We consider a family of integer linear programs in which the coefficients of the constraints and objective function are polynomials of an integer parameter t . For ℓ in \mathbb{Z}_+ , we define $f_\ell(t)$ to be the ℓ^{th} largest value of the objective function with multiplicity for the integer linear program at t . We prove that for all ℓ , f_ℓ is eventually quasi-polynomial; that is, there exists d and polynomials P_0, \dots, P_{d-1} such that for sufficiently large t , $f_\ell(t) = P_{d \pmod t}(t)$. Closely related to finding the ℓ^{th} largest value is describing the vertices of the convex hull of the feasible set. Clegg and Walker showed that if $R(t)$ is the convex hull of $\mathbf{v}_1(t), \dots, \mathbf{v}_k(t)$ where \mathbf{v}_i is a vector whose coordinates are in $\mathbb{Q}(u)$ and of size $O(u)$, then the vertices of the convex hull of the set of lattice points in $R(t)$ has eventually quasi-polynomial structure. We prove this without the $O(u)$ assumption.

1. INTRODUCTION

An integer program is the optimization of a certain objective function over the integers subject to certain constraints. Often in integer programming, the constraints and objective functions are linear functions of the indeterminates. This is known as integer linear programming.

Suppose that the indeterminates are $\mathbf{x} = (x_1, \dots, x_n)$. A program in canonical form has the following structure:

- 1) The objective function is $\mathbf{c}^T \mathbf{x}$ for $\mathbf{c} \in \mathbb{Z}^n$
- 2) The constraints are $\mathbf{x} \in \mathbb{Z}_{\geq 0}^n$ and $A\mathbf{x} \leq \mathbf{b}$ for $A \in \mathbb{Z}^{m \times n}$ and $\mathbf{b} \in \mathbb{Z}^m$. (In this paper, relations between vectors are coordinate-wise.)

A program in standard form has the same structure, except we use $A\mathbf{x} = \mathbf{b}$ instead of $A\mathbf{x} \leq \mathbf{b}$.

Parametric Integer Linear Programming (PILP) refers to considering a family of linear integer programs parametrized by a variable t i.e. the coefficients of the objective and/or constraints are functions of t . The optimum value of the objective function is a function of t (which we call the optimum value function), which leads us to questions about this function. Examples for the domain of t are the interval $[0, 1]$ or the positive integers. There are many algorithmic results on PILPs but few theoretical results on the properties of the optimum value function or other properties of the PILP.

Our main question concerns parametric integer linear programs in which all coefficients are integer polynomials in t and t ranges over the positive integers. In this paper, t is a positive integer. We prove that the optimum value function is eventually quasi-polynomial.

Definition 1.1. g is *eventually quasi-polynomial* (EQP) if its domain is a subset of \mathbb{Z} that contains sufficiently large integers and there exists a positive integer d and polynomials $P_0, \dots, P_{d-1} \in \mathbb{R}[u]$ such that for sufficiently large t , $g(t) = P_{t \pmod d}(t)$.

There are many results on EQPs ranging from the simple to the sophisticated. For example, if P_1, \dots, P_n are in $\mathbb{Z}[u]$, then $\gcd(P_1(t), \dots, P_n(t))$ and $\text{lcm}(P_1(t), \dots, P_n(t))$ are quasi-polynomials

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of t (even when the domain is all integers). If P is in $\mathbb{Q}(u)$, one can show that $\lfloor P(t) \rfloor$ is EQP; it is in fact implied by theorem 1.3 below. An early result on quasi-polynomials was by Ehrhart in [4].

Theorem 1.2. *Let P be a convex polytope whose vertices are rational vectors. For all t , let $f(t)$ be the number of lattice points in tP . Then f is quasi-polynomial (not just EQP).*

In fact, Macdonald showed that f is meaningful for negative integers[5]. The set tP is the real vector set (see subsection 1.1) of a PILP, so this theorem states that the size function of a certain class of PILPs is EQP. Chen, Li, and Sam showed that this is true for all PILPs[3]. We discuss their result later.

Woods proved that a certain class of parametric lattice point-counting problems defined by linear inequalities, boolean operations, and quantifiers has EQP behavior, even with more than one parameter[8]. The results in our paper are not true for more than one parameter.

A common theme among these results is that certain operations on polynomials of one or more integer parameters can form other integer-valued functions with domain \mathbb{Z}_+ such as a GCD function or a size function which are EQP. In some cases, the function may even be quasi-polynomial, with meaningful values for negative arguments. If the function is not integer-valued, like a quotient, then we do not expect anything. Our results follow this first pattern.

We study the ℓ^{th} largest value attained by the objective function with multiplicity. By this, we mean the ℓ^{th} largest value in the multiset obtained by evaluating the objective function on the set. We prove the following theorems, which also define the structure of a PILP in canonical or standard form. In these theorems, we require a certain boundedness condition which many reasonable PILPs satisfy.

Theorem 1.3 (Canonical Form). *Let n and m be positive integers. Let \mathbf{c} be in $\mathbb{Z}[u]^n$, A be in $\mathbb{Z}[u]^{m \times n}$, and \mathbf{b} be in $\mathbb{Z}[u]^m$.*

For all t , let $R(t) := \{\mathbf{x} \in \mathbb{R}^n \mid \mathbf{x} \geq \mathbf{0} \wedge A(t)\mathbf{x} \leq \mathbf{b}(t)\}$, the set of real vectors that satisfy all constraints except being integer vectors. Let $L(t) := R(t) \cap \mathbb{Z}^n$, the set of lattice points in $R(t)$. Assume that $R(t)$ is bounded for all t . For all positive integers ℓ , let $f_\ell(t)$ be the ℓ^{th} largest value of $\mathbf{c}^\top(t)\mathbf{x}$ for \mathbf{x} in $L(t)$ or $-\infty$ if $|L(t)| < \ell$.

Then for all ℓ , f_ℓ is eventually quasi-polynomial.

Remark. The constraints $\mathbf{x} \geq \mathbf{0}$ are not essential. See Corollary 3.2.2 below.

Theorem 1.4 (Standard Form). *Let n and m be positive integers. Let \mathbf{c} be in $\mathbb{Z}[u]^n$, A be in $\mathbb{Z}[u]^{m \times n}$, and \mathbf{b} be in $\mathbb{Z}[u]^m$.*

For all t , let $R(t) := \{\mathbf{x} \in \mathbb{R}^n \mid \mathbf{x} \geq \mathbf{0} \wedge A(t)\mathbf{x} = \mathbf{b}(t)\}$, the set of real vectors that satisfy all constraints except being integer vectors. Let $L(t) := R(t) \cap \mathbb{Z}^n$, the set of lattice points in $R(t)$. Assume that $R(t)$ is bounded for all t . For all positive integers ℓ , let $f_\ell(t)$ be the ℓ^{th} largest value of $\mathbf{c}^\top(t)\mathbf{x}$ for \mathbf{x} in $L(t)$ or $-\infty$ if $|L(t)| < \ell$.

Then for all ℓ , f_ℓ is eventually quasi-polynomial.

The constant function at $-\infty$ is to be interpreted as a polynomial. The canonical form and standard form are closely related, as we prove in the next section. Both forms of PILPs are considered because in general, it is easier to formulate a problem as a PILP in canonical form, but the reduction in Section 3 is more convenient in standard form.

Chen, Li, and Sam proved (as Theorems 1.1 and 2.1 in [3]), that for both the canonical and standard form, the size of $L(t)$ as a function of t is eventually quasi-polynomial, which is a generalization of Ehrhart's theorem. They used base t representations to reduce the problem to the case when A is in $\mathbb{Z}^{m \times n}$ and the coordinates of \mathbf{b} have degree at most 1. We show that the same idea applies when considering the ℓ^{th} largest value.

It is likely that in many cases, the components of the EQP f_ℓ , or the polynomials P_0, \dots, P_{d-1} shown in the definition, have the same degree. However, we do not discuss this in this paper. We

also do not find explicit bounds on the period of f_ℓ , or d from the definition, or an explicit integer N such that $f_\ell(t) = P_{t \pmod{d}}(t)$ for $t > N$.

The motivation for pursuing the results in this paper was a parametrized version of the Frobenius problem in which the arguments are integer valued polynomials. See [6]. Theorem 1.3 and related ideas may be useful in studying a variety of parametric combinatorics problems whose answers are suspected to be eventually quasi-polynomial such as those in [9].

Closely related to the idea of finding an optimum value is finding the vertices of the convex hull of $L(t)$ because the optimum value of a linear objective function is attained at some vertex of the convex hull of the feasible set. Calegari and Walker showed that if the vertices of $R(t)$ are of size $O(t)$, then the vertices of the convex hull of $L(t)$ have eventually quasi-polynomial structure[2]. See theorem 5.1. We show that this is true without the $O(t)$ assumption as a consequence of the base t idea.

1.1. Notation. In this paper, relations between vectors are coordinate-wise. The variables t and ℓ are always positive integers. The phrase $t \gg 0$ means “for sufficiently large t .” The constant function at $-\infty$ is to be interpreted as a polynomial. PILP means “parametric integer linear program.” EQP means “eventually quasi-polynomial” or “eventual quasi-polynomial.” Magnitude and distance refer to the Euclidean metric. By ℓ^{th} largest value, we mean the ℓ^{th} largest value with multiplicity.

By \mathbf{x} , we mean the vector (x_1, \dots, x_n) . A parametric inequality has the form $\mathbf{a}^\top(t)\mathbf{z} \leq b(t)$, where \mathbf{a} is a $\mathbb{Z}[u]$ -vector of the correct dimension, b is in $\mathbb{Z}[u]$, and \mathbf{z} refers to the vector of indeterminates e.g. \mathbf{x} , and a parametric equation is the same with equality.

We say that the PILP in canonical or standard form is defined by n, m, A, \mathbf{b} , and \mathbf{c} as in Theorems 1.3 and 1.4. We call $R(t)$ the real vector set of the PILP and $L(t)$ the lattice point set. We may also call these sets regions. $\{f_\ell\}_\ell$ or $\{f_\ell\}$ is the family of optimum value functions.

2. PROOF THAT THEOREM 1.4 IMPLIES THEOREM 1.3

The other direction is clear because one parametric equation can be written as two parametric inequalities.

Assume Theorem 1.4. To prove Theorem 1.3, we use the standard method of introducing slack variables. Let the PILP in canonical form be given by positive integers n and m , \mathbf{c} in $\mathbb{Z}[u]^n$, A in $\mathbb{Z}[u]^{m \times n}$, and \mathbf{b} in $\mathbb{Z}[u]^m$ which defines optimum value functions $\{f_\ell\}$, regions $R(t)$ and $L(t)$ which are bounded for all t . Consider the PILP in standard form with indeterminates $\mathbf{y} = (y_1, \dots, y_{n+m})$. Let $\mathbf{y}^1 = (y_1, \dots, y_n)$, $\mathbf{y}^2 = (y_{n+1}, \dots, y_{n+m})$, and let the constraints be $A(t)\mathbf{y}^1 + \mathbf{y}^2 = \mathbf{b}(t)$ and $y_i \in \mathbb{Z}_{\geq 0}$ for all i . The former set of constraints can easily be written as a parametric equation of \mathbf{y} . Let the objective function be $\mathbf{c}^\top(t)\mathbf{y}^1$, which can be written as a polynomial covector times \mathbf{y} . Let $R^*(t)$ and $L^*(t)$ be the regions defined by the second PILP and $\{f_\ell^*\}$ be the optimum value functions.

To use Theorem 1.4, we wish to show that $R^*(t)$ is bounded for each t . By assumption, $R(t)$ is bounded. As a subset of \mathbb{R}^n , $R(t)$ is bounded by some box $[-M, M]^n$ where M is constant (for fixed t). Let \mathbf{y} be in $R^*(t)$. The first n coordinates of \mathbf{y} lie in $R(t)$, so these coordinates lie in $[-M, M]$. When k is an integer in $[n+1, n+m]$, $y_k = (\mathbf{b}(t) - A(t)(y_1, \dots, y_n))_{k-n}$. For fixed t , the entries of $\mathbf{b}(t)$ and $A(t)$ are bounded in magnitude by some constant N . It follows that $|y_k| \leq N + nNM$ and that $R^*(t)$ is bounded for all t . By Theorem 1.4, for each ℓ , f_ℓ^* is EQP.

There is an obvious bijection between the lattice point sets: if (a_1, \dots, a_n) lies in $L(t)$, then (a_1, \dots, a_{n+m}) where a_k (for $n+1 \leq k \leq n+m$) is defined as $(\mathbf{b}(t) - A(t)(a_1, \dots, a_n))_{k-n}$ lies in $L^*(t)$. The inverse map is just ignoring the last m coordinates, and maps $L^*(t)$ to $L(t)$. Therefore, the two sets have the same size for all t . Evaluating the objective function commutes with the bijection by construction, so for all ℓ and t , $f_\ell(t) = f_\ell^*(t)$.

3. REDUCTION USING BASE t REPRESENTATIONS

In this section, we show that Theorem 1.4 can be reduced to the case of a PILP in canonical form in which the matrix has constant entries and the vector on the right hand side has entries which have degree at most 1. The main idea of this reduction is to express all indeterminates x_i in base t . This idea is taken from [3], in which the authors used this idea to show that the size of $L(t)$ is eventually quasi-polynomial. It is reproduced here with adjustments for considering optimum value functions.

Theorem 3.1 (Reduced canonical form). *Let n and m be positive integers. Let \mathbf{c} be in $\mathbb{Z}[u]^n$, A be in $\mathbb{Z}^{m \times n}$, and \mathbf{b} be in $(\mathbb{Z}u + \mathbb{Z})^m$.*

For all t , let $R(t) := \{\mathbf{x} \in \mathbb{R}^n \mid \mathbf{x} \geq \mathbf{0} \wedge A(t)\mathbf{x} \leq \mathbf{b}(t)\}$, the set of real vectors that satisfy all constraints except being integer vectors. Let $L(t) := R(t) \cap \mathbb{Z}^n$, the set of lattice points in $R(t)$. Assume that $R(t)$ is bounded for all t . For all ℓ , let $f_\ell(t)$ be the ℓ^{th} largest value of $\mathbf{c}^\top(t)\mathbf{x}$ for \mathbf{x} in $L(t)$ or $-\infty$ if $|L(t)| < \ell$.

Then for all ℓ , f_ℓ is eventually quasi-polynomial.

We prove Theorem 3.1 in the next section. The rest of this section is a proof that Theorem 3.1 implies Theorem 1.4.

Consider the PILP in standard form Q given by positive integers n and m , \mathbf{c} in $\mathbb{Z}[u]^n$, A in $\mathbb{Z}[u]^{m \times n}$, and \mathbf{b} in $\mathbb{Z}[u]^m$ which defines regions $R(t)$ and $L(t)$ and optimum value functions $\{f_\ell\}$. In order to express x_i in base t , we need that all coordinates in $L(t)$ are bounded by some power of t .

We use simple facts about convex polytopes. First, the extreme values of the coordinates occur at the vertices of the polytope. Second, given the hyperplane description of a convex polytope, or a description of a polytope as the intersection of finitely many closed half-spaces, each vertex is the unique intersection of the bounding hyperplanes on which it lies. This is based on the fact that a vertex of a polytope is not a nontrivial convex combination of other points of the polytope. Third, from linear algebra, among a set of hyperplanes in \mathbb{R}^n which intersect at a single point, some n of the hyperplanes intersect at a single point. The third fact is proven by showing that the normal vectors span \mathbb{R}^n . Then, some n of the dual vectors form a basis, and the corresponding hyperplanes intersect at a single point.

Proposition 3.2. *For a PILP in canonical form, there exists a positive integer r such that for $t \gg 0$, \mathbf{x} in $L(t)$, and all i , $0 \leq x_i < t^r$.*

Proof. For $i = 1, \dots, n$, and all t , let $P_i(t)$ be the hyperplane $x_i = 0$. In A , we ignore the rows which are identically 0 because the corresponding constraints ($0 \leq \mathbf{b}_k(t)$) are either eventually true or eventually false. In the former case, the constraint eventually does nothing, and in the latter case, $f(t)$ is eventually $-\infty$. For $i = 1, \dots, m$, let $P_{n+i}(t)$ be the hyperplane $(A(t)\mathbf{x})_i = \mathbf{b}(t)_i$. For some t , this may not actually be a hyperplane because all coefficients on the left hand side may be 0. Since at least one coefficient on the left hand side is a nonzero polynomial, $P_{n+i}(t)$ is a hyperplane for $t \gg 0$. Indeed, for $t \gg 0$, $P_{n+i}(t)$ is a hyperplane for all i .

Let S be the set of size n subsets of $\{1, \dots, n + m\}$. For each T in S , consider the system of parametric equations given by the parametric hyperplanes P_h for h in T . The coefficients of the left hand sides form a polynomial $n \times n$ matrix. If its determinant is the zero polynomial, then for all t , the n hyperplanes $P_h(t)$ for h in T do not meet at a single point. If its determinant is not the zero polynomial, then the inverse of the matrix exists for $t \gg 0$ and is a matrix of rational functions. The intersection of these n parametric hyperplanes is given by multiplying the inverted matrix by the vector of the right hand sides, so the intersection is a vector of rational functions, \mathbf{v}_T , in $\mathbb{R}(t)^n$.

A rational function of t is eventually bounded in magnitude by some power of t . Since n and $|S|$ are finite, there exists a positive integer r such that all coordinates of \mathbf{v}_T for all T (where the determinant is not identically 0) are bounded in magnitude by t^r for $t \gg 0$.

Fix $t \gg 0$ so that the hyperplanes $P_1(t), \dots, P_{n+m}(t)$ are nondegenerate and all coordinates of \mathbf{v}_T for all T are bounded in magnitude by t^r . If $R(t)$ is empty, then we vacuously have that $0 \leq x_i < t^r$ for all \mathbf{x} in $L(t)$ and all i . If $R(t)$ is nonempty, it is bounded by assumption, so it is a nonzero closed polytope. The extreme values of the coordinates of $R(t)$ occur at its vertices. Each vertex of $R(t)$ is the intersection of n of the hyperplanes $P_1(t), \dots, P_{n+m}(t)$ which intersect at a single point, so the determinant as described above is not identically zero, and the vertex is one of the vectors $\mathbf{v}_T(t)$ for T in S . Each coordinate of $\mathbf{v}_T(t)$ is bounded in magnitude by t^r for $t \gg 0$. This is true for all vertices, so for any point in $L(t)$, which is a subset of $R(t)$, each coordinate is bounded in magnitude by t^r for $t \gg 0$. Lastly, each coordinate is nonnegative because the PILP is in canonical form. \square

Corollary 3.2.1. *For a PILP in standard form, there exists an integer r such that for $t \gg 0$, \mathbf{x} in $L(t)$, and all i , $0 \leq x_i < t^r$.*

Corollary 3.2.2. *Theorem 1.3 implies the following: Let n and m be positive integers. Let \mathbf{c} be in $\mathbb{Z}[u]^n$, A be in $\mathbb{Z}[u]^{m \times n}$, and \mathbf{b} be in $\mathbb{Z}[u]^m$.*

For all t , let $R(t) := \{\mathbf{x} \in \mathbb{R}^n \mid A(t)\mathbf{x} \leq \mathbf{b}(t)\}$, the set of real vectors that satisfy all constraints except being integer vectors. Let $L(t) := R(t) \cap \mathbb{Z}^n$, the set of lattice points in $R(t)$. Assume that $R(t)$ is bounded for all t . For all positive integers ℓ , let $f_\ell(t)$ be the ℓ^{th} largest value of $\mathbf{c}^\top(t)\mathbf{x}$ for \mathbf{x} in $L(t)$ or $-\infty$ if $|L(t)| < \ell$.

Then for all ℓ , f_ℓ is eventually quasi-polynomial.

Proof. Assume Theorem 1.3. By the proof of Proposition 3.2, there exists a positive integer r such that each coordinate of $R(t)$ is bounded in magnitude by t^r for $t \gg 0$, say for $t > N$.

We can form another PILP Q' with regions $R'(t), L'(t)$, etc. and the same objective function such that $R'(t)$ is $R(t)$ translated by (t^r, \dots, t^r) (n coordinates). This is possible with polynomial A' and \mathbf{b}' because t^r is a polynomial. We also add the constraints $N + 1 \leq t$ and $\mathbf{x} \geq \mathbf{0}$ to Q' . By the definition of N , the constraints $\mathbf{x} \geq \mathbf{0}$ is implied by the others. Since $R(t)$ is bounded, $R'(t)$ is bounded, and it lies in the first orthant. By Theorem 1.3, f'_ℓ is EQP for all ℓ .

The translation by (t^r, \dots, t^r) increases the objective function by $\mathbf{c}^\top(t^r, \dots, t^r)$, a polynomial. For $t > N$ such that $f_\ell(t)$ is finite, $f_\ell(t) = f'_\ell(t) - \mathbf{c}^\top(t^r, \dots, t^r)$. For $t > N$ such that $f'_\ell(t) = -\infty$, $f_\ell(t) = -\infty$. A polynomial with range $\{-\infty\} \cup \mathbb{Z}$ is either the constant $-\infty$ or integer valued. From this fact and the fact that the EQP property only depends on large arguments, it easily follows that f_ℓ is EQP for all ℓ . \square

Let r be the positive integer for Q , which exists by the above proposition. We define several PILPs Q_α , each with r times as many indeterminates, which correspond to the coefficients of x_i in base t .

For each Q_α , let the indeterminates be $y_{i,j}$ for all i, j with $1 \leq i \leq n, 1 \leq j \leq r$, the objective function be $\sum_{i=1}^n c_i(t) \sum_{j=1}^r y_{i,j} t^{j-1}$ (which can be written as a polynomial covector times \mathbf{y}), the regions be $R_\alpha(t)$ and $L_\alpha(t)$, and the optimum value functions be $\{f_{\ell,\alpha}\}$. Here, \mathbf{y} refers to all $y_{i,j}$ as a vector. The $n \times r$ indices can be ordered as $\{1, 2, \dots, nr\}$, but it is more convenient to leave the indices as is.

Each Q_α has the usual constraints $y_{i,j} \geq 0$ and $y_{i,j} \in \mathbb{Z}$ along with $y_{i,j} \leq t - 1$.

The Q_α correspond to the various ways that carrying, positive or negative, can occur in the various component equations of $A(t)\mathbf{x} = \mathbf{b}(t)$ in base t .

Let

$$K := \max(r - 1 + \max_{i,j} \deg A_{ij}, \max_i \deg \mathbf{b}_i).$$

For a polynomial $P(u) = \sum_{h=0}^{h_{max}} a_h u^h$, let $\|P\|_1 = \sum_{h=0}^{h_{max}} |a_h|$, the sum of the absolute values of the coefficients. Write A_{ij} as $\sum_{h=0}^{K+1-r} A_{ij,h} t^h$ and \mathbf{b}_i as $\sum_{h=0}^K \mathbf{b}_{i,h} t^h$. Let

$$N := 1 + K \sum_{i,j} \|A_{ij}\|_1 + \sum_i \|\mathbf{b}_i\|_1.$$

Let

$$S := \{-N, -N+1, \dots, N\}^{m \times K}.$$

It only matters to us that N and S are finite. The Q_α are indexed by α in S . The remaining constraints of Q_α (in addition to $0 \leq y_{i,j} \leq t-1$) are

$$(1) \quad \sum_{i=1}^n \sum_{\substack{0 \leq h_1 \leq K+1-r \\ 0 \leq h_2 \leq r-1 \\ h_1+h_2=h}} A_{ij,h_1} y_{i,h_2+1} = \mathbf{b}_{j,h} + \alpha_{j,h+1}t - \alpha_{j,h},$$

for all j, h with $1 \leq j \leq m$ and $0 \leq h \leq K$. For all j , $\alpha_{j,K+1}$ and $\alpha_{j,0}$ are defined to be 0.

These constraints can be written in the form required in Theorem 3.1, meaning each constraint is of the form $\sum_{i,j} a_{i,j} y_{i,j} \leq a_1 t + a_0$, where all $a_{i,j}$, a_1 , and a_0 are integers. The constraints $y_{i,j} \leq t-1$ are also of this form. (It is possible for many of the sets $L_\alpha(t)$ to be empty.) The constraints $0 \leq y_{i,j} \leq t-1$ imply that $R(t)$ is bounded for all t . By Theorem 3.1, each $f_{\ell,\alpha}$ is eventually quasi-polynomial.

Proposition 3.3. *For all $t > 2N$ and ℓ , $f_\ell(t)$ is the ℓ^{th} largest value among the multiset*

$$\{f_{m,\alpha}(t) | 1 \leq m \leq \ell, \alpha \in S\}.$$

Proof. Fix $t > 2N$.

The main ideas are that the sets $L_\alpha(t)$ for α in S are disjoint and the correspondence $x_i = \sum_{j=1}^r y_{i,j} t^{j-1}$ is a bijection between $L(t)$ and $\sqcup_{\alpha \in S} L_\alpha(t)$ which commutes with evaluating the respective objective function.

Consider a point in $L_\alpha(t)$ for some α in S with coordinates $\{y_{i,j}\}$. Each $y_{i,j}$ is an integer in $[0, t-1]$. For each i , let $x_i = \sum_{j=1}^r y_{i,j} t^{j-1}$, which is a nonnegative integer.

The vector \mathbf{y} satisfies (2) for all j, h . Multiplying (2) by t^h gives

$$(2) \quad \sum_{i=1}^n \sum_{\substack{0 \leq h_1 \leq K+1-r \\ 0 \leq h_2 \leq r-1 \\ h_1+h_2=h}} A_{ij,h_1} y_{i,h_2+1} t^h = \mathbf{b}_{j,h} t^h + \alpha_{j,h+1} t^{h+1} - \alpha_{j,h} t^h.$$

For fixed j , adding (2) as h ranges from 0 to K gives

$$\begin{aligned} \sum_{i=1}^n \sum_{h=0}^K \sum_{\substack{0 \leq h_1 \leq K+1-r \\ 0 \leq h_2 \leq r-1 \\ h_1+h_2=h}} A_{ij,h_1} y_{i,h_2+1} t^h &= \sum_{h=0}^K \mathbf{b}_{j,h} t^h. \\ \sum_{i=1}^n A_{ij}(t) \sum_{l_2=0}^{r-1} y_{i,h_2+1} t^{h_2} &= \mathbf{b}_j(t). \\ (A\mathbf{x})_j &= \mathbf{b}_j(t). \end{aligned}$$

Therefore, \mathbf{x} lies in $L(t)$.

Conversely, consider a point \mathbf{x} in $L(t)$. By the definition of r , each x_i is an integer in $[0, t^r)$, so there exist $y_{i,j}$ for $1 \leq i \leq n, 1 \leq j \leq r$ such that each $y_{i,j}$ is an integer in $[0, t-1]$ and $x_i = \sum_{j=1}^r y_{i,j} t^{j-1}$. $y_{i,j}$ are the digits of x_i . This is the inverse of the map from \mathbf{y} to \mathbf{x} given above.

Denote the coefficients of the elements of A and \mathbf{b} as before. Let α be the m by K matrix of integers whose (j, h_0) entry is

$$\left(\sum_{h=h_0}^K \mathbf{b}_{j,h} t^{h-h_0} \right) - \left(\sum_{i=1}^n \sum_{h=h_0}^K \sum_{\substack{0 \leq h_1 \leq K+1-r \\ 0 \leq h_2 \leq r-1 \\ h_1+h_2=h}} A_{ij,h_1} y_{i,h_2+1} t^{h-h_0} \right),$$

for $1 \leq j \leq m, 1 \leq h_0 \leq K$. Since $(A\mathbf{x})_j = \mathbf{b}_j(t)$, this entry also equals

$$(3) \quad \left(\sum_{i=1}^n \sum_{h=0}^{h_0-1} \sum_{\substack{0 \leq h_1 \leq K+1-r \\ 0 \leq h_2 \leq r-1 \\ h_1+h_2=h}} A_{ij,h_1} y_{i,h_2+1} t^{h-h_0} \right) - \left(\sum_{h=0}^{h_0-1} \mathbf{b}_{j,h} t^{h-h_0} \right).$$

The former representation shows that the entry is an integer, and the latter gives us a bound.

$$|\alpha_{j,h_0}| \leq \left(\sum_{i=1}^n \sum_{h=0}^{h_0-1} \sum_{\substack{0 \leq h_1 \leq K+1-r \\ 0 \leq h_2 \leq r-1 \\ h_1+h_2=h}} |A_{ij,h_1} y_{i,h_2+1}| t^{-1} \right) + \left(\sum_{h=0}^{h_0-1} |\mathbf{b}_{j,h}| t^{-1} \right),$$

where $t^{h-h_0} \leq t^{-1}$ over the relevant ranges. By definition, $|\mathbf{b}_j| < N < t$ and $0 \leq y_{i,h_2+1} < t$, so

$$\begin{aligned} |\alpha_{j,h_0}| &\leq \left(\sum_{i=1}^n \sum_{h=0}^{h_0-1} \sum_{\substack{0 \leq h_1 \leq K+1-r \\ 0 \leq h_2 \leq r-1 \\ h_1+h_2=h}} |A_{ij,h_1}| \right) + 1. \\ |\alpha_{j,h_0}| &\leq \left(\sum_{i=1}^n \sum_{h=0}^{h_0-1} |A_{ij}| \right) + 1 \leq 1 + K \sum_{i=1}^n |A_{ij}| \leq N. \end{aligned}$$

Therefore, $\alpha \in S$.

From (3), we have

$$\begin{aligned} &\left(\sum_{i=1}^n \sum_{h=0}^{h_0-1} \sum_{\substack{0 \leq h_1 \leq K+1-r \\ 0 \leq h_2 \leq r-1 \\ h_1+h_2=h}} A_{ij,h_1} y_{i,h_2+1} t^h \right) - \left(\sum_{h=0}^{h_0-1} \mathbf{b}_{j,h} t^h \right) = \alpha_{j,h_0} t^{h_0}. \\ &\left(\sum_{i=1}^n \sum_{h=0}^{h_0} \sum_{\substack{0 \leq h_1 \leq K+1-r \\ 0 \leq h_2 \leq r-1 \\ h_1+h_2=h}} A_{ij,h_1} y_{i,h_2+1} t^h \right) - \left(\sum_{h=0}^{h_0} \mathbf{b}_{j,h} t^h \right) = \alpha_{j,h_0+1} t^{h_0+1}. \end{aligned}$$

Subtracting the former equation from the latter, we get

$$\left(\sum_{i=1}^n \sum_{\substack{0 \leq h_1 \leq K+1-r \\ 0 \leq h_2 \leq r-1 \\ h_1+h_2=h_0}} A_{ij,h_1} y_{i,h_2+1} t^{h_0} \right) - \left(\mathbf{b}_{j,h_0} t^{h_0} \right) = \alpha_{j,h_0+1} t^{h_0+1} - \alpha_{j,h_0} t^{h_0}.$$

Dividing by t^{h_0} and rearranging gives (2), which shows that \mathbf{y} is in $L_\alpha(t)$.

The sets $L_\alpha(t)$ are disjoint because $t > 2N$ (see (1)). The above correspondences between \mathbf{x} and its digits \mathbf{y} are inverses, so it gives a bijection between $L(t)$ and the disjoint union of the sets $L_\alpha(t)$. By construction, evaluating the respective objective function commutes with the bijection. It easily follows that the ℓ^{th} largest value of the objective function in $L(t)$ is given by the ℓ^{th} largest value among the multiset union of the ℓ largest values of the objective function in all sets $L_\alpha(t)$. (If $|L_\alpha(t)| < \ell$, our convention is that the multiset of the ℓ largest values includes $-\infty$ $\ell - |L_\alpha(t)|$ times.) \square

Finally, we combine all the $f_{\ell,\alpha}$ with the following proposition.

Proposition 3.4. *Let m and ℓ be positive integers and f_1, \dots, f_m be eventually quasi-polynomial with range $\{-\infty\} \cup \mathbb{Z}$. For all t , let $f(t)$ be the ℓ^{th} largest value among the multiset $\{f_1(t), \dots, f_m(t)\}$. Then f is eventually quasi-polynomial.*

Proof. Because m is finite, there exists a common period for f_1, \dots, f_m , or an integer d such that there exist polynomials $P_{i,j}$ for all integers i and j with $1 \leq i \leq m$ and $0 \leq j \leq d-1$ such that for $t \gg 0$ and all i , $f_i(t) = P_{i,(t \bmod d)}(t)$. A polynomial with range $\{-\infty\} \cup \mathbb{Z}$ is either the constant $-\infty$ or integer valued. If $f(t)$ restricted to each residue class $(\bmod d)$ is EQP, then f is EQP. Since we only care about large t , it suffices to prove this proposition in the case when f_1, \dots, f_m are all polynomials (or the constant $-\infty$), so assume that this is the case.

If less than ℓ of the polynomials f_1, \dots, f_m are integer valued, then for all t , the multiset $\{f_1(t), \dots, f_m(t)\}$ contains less than ℓ integers, so $f(t) = -\infty$. If at least ℓ of the polynomials f_1, \dots, f_m are integer valued, say f_1 through f_k , then for all t , $f(t)$ is the ℓ^{th} largest value among the integers $f_1(t), \dots, f_k(t)$. It is known that there exists a permutation of the polynomials $f_1, \dots, f_k, f_{\sigma(1)}, \dots, f_{\sigma(k)}$, such that for $t \gg 0$, $f_{\sigma(1)}(t) \geq \dots \geq f_{\sigma(k)}(t)$, so $f(t) = f_{\sigma(\ell)}(t)$ for $t \gg 0$, as desired. \square

This completes the proof that Theorem 1.4 follows from Theorem 3.1

4. PROOF OF THEOREM 3.1

We now allow the case $n = 0$ in Theorem 3.1. We prove Theorem 3.1 first in the case $n = 0$. The idea is to prove the remaining cases by strong induction on $\ell + n$.

Suppose that $n = 0$. $f_\ell(t) = 0$ if $\ell = 1$ and $L(t)$ is nonempty, and $f_\ell(t) = -\infty$ otherwise. $L(t)$ is nonempty if and only if $\mathbf{0} \leq \mathbf{b}(t)$. Since the elements of \mathbf{b} are polynomials, this inequality is either eventually true or eventually false (as t goes to ∞). This corresponds to $f_1(t) = 0$ eventually or $f_1(t) = -\infty$ eventually, respectively, as desired.

We now prove some propositions needed for the induction. Let Q be a PILP in canonical form given by positive integers n_0 and m , \mathbf{c} in $\mathbb{Z}[u]^{n_0}$, A in $\mathbb{Z}^{m \times n_0}$, and \mathbf{b} in $(\mathbb{Z}u + \mathbb{Z})^m$ which defines regions $R(t)$ and $L(t)$ and optimum value function f_{ℓ_0} .

Proposition 4.1. *For all t and ℓ , $f_\ell(t)$ equals the ℓ^{th} largest value of $\mathbf{c}^\top(t)\mathbf{x}$ over all \mathbf{x} in $L(t)$ which are less than ℓ away from some bounding hyperplane of $R(t)$ (in the Euclidean metric).*

Proof. Fix t and ℓ .

Case 1: $|L(t)| < \ell$. Then $f_\ell(t) = -\infty$, and the ℓ^{th} largest value of $\mathbf{c}^\top(t)\mathbf{x}$ over all \mathbf{x} in $L(t)$ which are less than ℓ away from some bounding hyperplane of $R(t)$ is naturally equal to $-\infty$. See Proposition 4.2 for specifics.

Case 2: $|L(t)| \geq \ell$. Then $f_\ell(t)$ is a real number.

Case 2.1: $\mathbf{c}^\top(t) = \mathbf{0}$. The objective function is constant. To prove the proposition in this case, we need to show that at least ℓ distinct points in $L(t)$ are less than ℓ away from some bounding hyperplane of $R(t)$. Successively choose ℓ points $\mathbf{v}_1, \dots, \mathbf{v}_\ell$ with minimal x_1 coordinate from $L(t)$ without replacement, which is possible since $|L(t)| \geq \ell$. Let \mathbf{e}_1 be the unit vector in the first coordinate. For each \mathbf{v}_i , $\mathbf{v}_i - \mathbf{e}_1, \dots, \mathbf{v}_i - \ell\mathbf{e}_1$ cannot all be in $L(t)$, or else \mathbf{v}_i could not be picked in this process. Therefore, for $i = 1, \dots, \ell$, \mathbf{v}_i is less than ℓ away from the boundary of $R(t)$, as desired.

Case 2.2: $\mathbf{c}^\top(t) \neq \mathbf{0}$. To prove the proposition, it suffices to show that for $k = 1, \dots, \ell$, the k^{th} largest value does not occur at any point of $L(t)$ which is at least ℓ away from the boundary of $R(t)$. Suppose, for the sake of contradiction, that for $k \leq \ell$, the k^{th} largest value occurs at a point \mathbf{v} in $L(t)$ which is at least ℓ away from the boundary of $R(t)$. Since the objective function is linear but not constant, it is greater at some lattice point 1 away from \mathbf{v} , say \mathbf{w} , than it is at \mathbf{v} . By assumption, for $s = 1, \dots, \ell$, $\mathbf{v} + s(\mathbf{w} - \mathbf{v})$ lies in $R(t)$ hence in $L(t)$, and

$$\mathbf{c}^\top(t)(\mathbf{v} + s(\mathbf{w} - \mathbf{v})) > \mathbf{c}^\top(t)(\mathbf{v}),$$

which contradicts that the k^{th} largest value occurs at \mathbf{v} . Therefore, the assumption was wrong, as desired. \square

By standard vector calculations, a lattice point less than ℓ_0 away from the hyperplane $\sum_{i=1}^{n_0} a_i x_i = a_0$ lies on a hyperplane $\sum_{i=1}^{n_0} a_i x_i = a_0 + j$, where

$$|j| < \ell_0 \sqrt{\sum_{i=1}^n a_i^2}.$$

If a_0, \dots, a_n are all integers, then j is an integer (and there are finitely many hyperplanes to consider).

We can write the inequalities $\mathbf{x} \geq 0$ as extra rows for A and \mathbf{b} in an obvious way. The constraints of Q are now $A\mathbf{x} \leq \mathbf{b}(t)$. We ignore the rows of A which are all 0 because these rows define inequalities that are either true for sufficiently large t or false for sufficiently large t . In the former case, we can ignore the row for $t \gg 0$, and in the latter, $f_{\ell_0}(t)$ is eventually $-\infty$. We redefine m so that A has m rows.

For all i with $1 \leq i \leq m$, let

$$c_i := \left\lceil \ell_0 \sqrt{\sum_{j=1}^n A_{ij}^2} \right\rceil.$$

We define c_i PILPs which correspond to the parametric hyperplanes near to and parallel to the parametric hyperplane $(A\mathbf{x})_i \leq \mathbf{b}(t)_i$. For $k = 0, 1, \dots, c_i - 1$ let $Q_{i,k}$ be the PILP in canonical form which is our original PILP with the same objective function and constraints with some extra constraints:

$$\left(\sum_{j=1}^{n_0} A_{ij} x_j \geq b_i(t) - k \right) \wedge \left(- \sum_{j=1}^{n_0} A_{ij} x_j \geq -b_i(t) + k \right)$$

and for $h = 1, \dots, i - 1$,

$$\sum_{j=1}^{n_0} A_{hj} x_j \leq b_h(t) - c_h.$$

Note that b_i and b_h have degree at most 1. Let $Q_{i,k}$ define regions $R_{i,k}(t)$ and $L_{i,k}(t)$ and optimum value functions $\{f_{\ell,i,k}\}$. (As before, it is possible for many of the sets $L_{i,k}(t)$ to be empty.)

Proposition 4.2. *For all t , $f_{\ell_0}(t)$ equals the ℓ_0^{th} largest value among the multiset*

$$\{f_{\ell,i,k}(t) | 1 \leq \ell \leq \ell_0, 1 \leq i \leq m, 0 \leq k < c_i\}.$$

Proof. By construction, for fixed t , the regions $L_{i,k}(t)$ are disjoint. For fixed t , consider \mathbf{x} in $L(t)$ which is less than ℓ_0 away from the boundary of $R(t)$. There are finitely many bounding hyperplanes, so there exists a minimal i such that \mathbf{x} is less than ℓ_0 away from the hyperplane given by the i^{th} row. Then \mathbf{x} lies in $L_{i,k}(t)$, where k is the distance from \mathbf{x} to the i^{th} hyperplane (at t) times $(\sum_j A_{ij}^2)^{1/2}$, (which is less than c_i). This shows that the disjoint union of $L_{i,k}(t)$ for $1 \leq i \leq m$ and $0 \leq k < c_i$ is the set of all points in $L(t)$ which are less than ℓ_0 away from a bounding hyperplane of $R(t)$. Combining this fact with Proposition 4.1 shows that $f_{\ell_0}(t)$ is the ℓ_0^{th} largest value in the union of the multisets of the ℓ_0 largest values from all $L_{i,k}(t)$. \square

Proposition 4.3. *Assume that ℓ_0, n_0 are positive integers such that Theorem 3.1 is true for all positive integers ℓ, n such that $\ell + n < \ell_0 + n_0$. Fix ℓ, i , and k such that $\ell \leq \ell_0, 1 \leq i \leq m, 0 \leq k < c_i$. $f_{\ell,i,k}$ is EQP.*

Proof. The idea is that since $R_{i,k}(t)$ lies on a hyperplane, $Q_{i,k}$ is like some PILP with $n_0 - 1$ indeterminates. Intuitively, $f_{\ell,i,k}$ should be an eventual quasi-polynomial plus the ℓ^{th} optimum value function for this other PILP.

For all t , $R_{i,k}(t)$ lies on the hyperplane $\sum_{j=1}^{n_0} A_{i,j} x_j = b_i(t) - k$; call this hyperplane $P(t)$. Let $d = \gcd(A_{i,1}, \dots, A_{i,n_0})$, which can be written as $\sum_{h=1}^{n_0} \beta_h A_{i,h}$ for integers β_h . $P(t)$ contains lattice points (not necessarily in $L_{i,k}(t)$) if and only if $b_i(t) - k$ is a multiple of d . Since b_i has degree at most 1, this occurs either never or periodically i.e. there exist positive integers p and q such that $P(t)$ contains lattice points if and only if $t \equiv p \pmod{q}$. In the former case (“never”), $f_{\ell,i,k}(t) = -\infty$, and we are done. Therefore, assume the “periodic” case; then $f_{\ell,i,k}(t) = -\infty$ when $t \not\equiv p \pmod{q}$.

Suppose that $t \equiv p \pmod{q}$. Let $g(t)$ be the polynomial $\frac{1}{d}(b_i(t) - k)$, which has degree at most 1 and is an integer for these t . The lattice point $(g(t)\beta_1, \dots, g(t)\beta_{n_0})$ lies in $P(t)$. The lattice points in $P(t)$ translated by $-(g(t)\beta_1, \dots, g(t)\beta_{n_0})$ form an integer lattice U , independent of t , which spans an $n_0 - 1$ dimensional space. Therefore there exist independent vectors in \mathbb{Z}^{n_0} , $\mathbf{e}_1, \dots, \mathbf{e}_{n_0-1}$, such that $\varphi_t : \mathbb{Z}^{n_0-1} \rightarrow \mathbb{Z}^{n_0}$ given by

$$\varphi_t(\mathbf{y}) = (g(t)\beta_1, \dots, g(t)\beta_{n_0}) + \sum_{h=1}^{n_0-1} y_h \mathbf{e}_h$$

is injective with image $\mathbb{Z}^{n_0} \cap P(t)$.

Define a new PILP Q' not in canonical form with indeterminates $\mathbf{y} = (y_1, \dots, y_{n_0-1})$ (which correspond to coefficients of the \mathbf{e}_h), regions $R'(t)$ and $L'(t)$, and optimum value functions $\{f'_\ell\}$. Each constraint of $Q_{i,k}$ has the form $\mathbf{w}^\top \mathbf{x} \leq W(t)$, where \mathbf{w} is an integer vector and W is in $\mathbb{Z}u + \mathbb{Z}$. For each such constraint, Q' has the constraint

$$\mathbf{w}^\top \left((g(t)\beta_1, \dots, g(t)\beta_{n_0}) + \sum_{h=1}^{n_0-1} y_h \mathbf{e}_h \right) \leq W(t),$$

and Q' has no other constraints. Since W and g are rational polynomials of degree at most 1, the constraints of Q' can be equivalently written as parametric inequalities where the right hand side

has degree at most 1 (and the coefficients are *integers*). The objective function, which we denote by $c(\mathbf{y})$, is

$$\mathbf{c}^\top(t) \left((g(t)\beta_1, \dots, g(t)\beta_{n_0}) + \sum_{h=1}^{n-1} y_h \mathbf{e}_h \right),$$

which can be written as a rational polynomial covector times \mathbf{y} plus a rational polynomial. Therefore, for some Z in $\mathbb{Z}[u]$ and positive integer z , $zc(\mathbf{y}) - Z(u)$ is an integer polynomial covector times \mathbf{y} .

By construction, for each t which is $p \pmod{q}$, the map φ_t defined above maps $R'(t)$ to $R_{i,k}(t)$. The map φ_t is a bijection between these two sets because φ_t is a bijection between \mathbb{Z}^{n_0-1} and $\mathbb{Z}^{n_0} \cap P(t)$. The bijection commutes with evaluating the respective objective function, so $f_{\ell,i,k}(t) = f'_\ell(t)$. When $t \not\equiv p \pmod{q}$, φ_t is not a bijection, so $f_{\ell,i,k}(t)$ and $f'_\ell(t)$ are not related. For all t , $R'(t)$ is bounded because it is contained in some affine transformation of $R_{i,k}(t)$.

Using the same argument as in Proposition 3.2, the candidate vertices of $R'(t)$ are found by multiplying the inverse of an integer matrix (whenever this is invertible) by a vector of polynomials of degree at most 1, which equals another vector of polynomials of degree at most 1. Therefore, there exists a positive integer K such that each coordinate of $R'(t)$ is bounded in magnitude by $Kt + K$. Translate the constraints of Q' by $(Kt + K, \dots, Kt + K)$ to form a new PILP Q'' with the same indeterminates and objective function. Form yet another PILP Q''' with the same indeterminates and constraints with objective function $zc - Z(u)$, which is multiplication by an integer polynomial covector. Let $\{f''_\ell\}$ be the family of optimum value functions for Q'' , etc.

$R'''(t)$ is bounded for all t , and by the definition of K , $R'''(t)$ lies in the first orthant, so we can write Q''' in canonical form by adding the constraints $\mathbf{y} \geq \mathbf{0}$ without changing its optimum value functions. Q''' has $n_0 - 1$ indeterminates, the right hand sides have degree at most 1, and $\ell \leq \ell_0$. Either the hypotheses of this proposition (if $n_0 > 1$) or the $n = 0$ case (if $n_0 = 1$) applies to f''_ℓ of Q''' , telling us that f''_ℓ is EQP.

For t such that $f''_\ell(t)$ is finite, $zf''_\ell(t) - Z(t) = f'''_\ell(t)$. For t such that $f''_\ell(t) = -\infty$, $f'''_\ell(t) = -\infty$. As in Proposition 3.4, a polynomial with range $\{-\infty\} \cup \mathbb{Z}$ is either the constant $-\infty$ or integer valued. This easily implies that f''_ℓ is EQP.

As in Corollary 3.2.2, the translation from Q' to Q'' adds a polynomial to the optimum value functions or takes $-\infty$ to $-\infty$. Therefore, f'_ℓ is EQP. Restricting an EQP to the residue class $p \pmod{q}$ and replacing the other outputs by $-\infty$ gives another EQP, so $f_{\ell,i,k}$ is EQP, as desired. \square

Proof of Theorem 3.1. We have already established the $n = 0$ case. We use strong induction on $\ell + n$. Let ℓ_0, n_0 be positive integers and assume that Theorem 3.1 is true for all positive integers ℓ, n such that $\ell + n < \ell_0 + n_0$. There is no base case of $\ell_0 + n_0$ because the case $n = 0$ acts as the base case.

By Proposition 4.2, $f_{\ell_0}(t)$ equals the ℓ_0^{th} largest value among the multiset

$$\{f_{\ell,i,k}(t) | 1 \leq \ell \leq \ell_0, 1 \leq i \leq m, 0 \leq k < c_i\}.$$

By Proposition 4.3, $f_{\ell,i,k}$ is EQP for $\ell \leq \ell_0, 1 \leq i \leq m$, and $0 \leq k < c_i$. Proposition 3.4 shows that f_{ℓ_0} is EQP, which shows that Theorem 3.1 is true for the case $\ell = \ell_0$ and $n = n_0$, which completes the induction. \square

Remark. Theorems 1.3 and 1.4 are not true for more than one parameter. For example, it is easy to form a PILP with parameters t_1 and t_2 whose optimum value function is $\max(t_1, t_2)$; this is not considered EQP.

In Theorems 1.3 and 1.4, one cannot replace ℓ with a polynomial. For example, consider the PILP with indeterminates x_1 and x_2 where $R(t)$ is the region $x_1 \geq 0, x_2 \geq 0, x_1 + x_2 \leq t$ and

$\mathbf{c} = (1, 0)$. $f(t) - t \sim \sqrt{2t}$ because roughly speaking, there are about t points in the top $\sqrt{2t}$ rows of $L(t)$. $t \mapsto f(t) - t$ is not EQP, being asymptotic to $\sqrt{2t}$, so neither is f .

5. THE CONVEX HULL OF THE LATTICE POINT SET

In this section, we study the convex hull of the lattice point set of a PILP and resolve a conjecture by Calegari and Walker. This convex hull is closely related to the first optimum value function of this PILP because the optimum value of a linear objective function occurs at a vertex of this convex hull.

In 2011, Calegari and Walker proved the following as Theorem 3.5 in [2].

Theorem 5.1. *For $1 \leq i \leq k$, let \mathbf{v}_i be in $\mathbb{Q}(u)^n$ of size $O(u)$, and for all t , let $R(t)$ be the convex hull of $\mathbf{v}_1(t), \dots, \mathbf{v}_k(t)$. Then there exists a positive integer d such that for $t \gg 0$ and t restricted to a single residue class $(\text{mod } d)$, there exist $\mathbf{w}_1, \dots, \mathbf{w}_z$ in $\mathbb{Q}[u]^n$ such that the convex hull of the set of lattice points in $R(t)$ has vertices $\mathbf{w}_1(t), \dots, \mathbf{w}_z$.*

The number of vertices of the convex hull of the lattice point set, z , need not be the same for each residue class or even positive. They proved this theorem by showing that the vertices are of bounded distance from the bounding hyperplanes as n goes to ∞ , which is similar to our argument in section 3. They conjectured that this theorem is still true for parametric vertices \mathbf{v}_i which are not of size $O(u)$ (see 3.9 in their paper), which we now prove.

Theorem 5.2. *For $1 \leq i \leq k$, let \mathbf{v}_i be in $\mathbb{Q}(u)^n$, and for all t , let $R(t)$ be the convex hull of $\mathbf{v}_1(t), \dots, \mathbf{v}_k(t)$. Then there exists a positive integer d such that for $t \gg 0$ and t restricted to a single residue class $(\text{mod } d)$, there exist $\mathbf{w}_1, \dots, \mathbf{w}_z$ in $\mathbb{Q}[u]^n$ such that the convex hull of the set of lattice points in $R(t)$ has vertices $\mathbf{w}_1(t), \dots, \mathbf{w}_z$.*

Proposition 5.3. *To prove Theorem 5.2, it suffices to show that the vertices of the convex hull of the lattice point set of a PILP in standard form has eventually quasi-polynomial structure. (See the second sentence of Theorem 5.2.)*

Proof. Let $\mathbf{v}_1, \dots, \mathbf{v}_k$ be in $\mathbb{Q}(u)^n$. Rational functions are bounded by polynomials for $t \gg 0$. Therefore, we can translate these parametric vertices by some polynomial vector so that $R^*(t)$ lies in the first orthant for $t \gg 0$. One can show that $R(t)$ coincides with the real vector set of some PILP in canonical form. See section 2.1 of [3]. Since $R^*(t)$ and $R(t)$ are translations by a polynomial vector, it suffices to show that the vertices of the convex hull of the lattice point set of a PILP in canonical form has eventually quasi-polynomial structure.

The bijection between the lattice point sets of a PILP in canonical form and a PILP in standard form from section 2 extends to a bijective affine transformation (the codomain is the hyperplane of the PILP in standard form). Bijective affine transformations preserve convex combinations and send polynomial vectors to polynomial vectors. Therefore, the vertices of the lattice point sets of two PILPs are in bijection by this same affine transformation, and it suffices to consider a PILP in standard form. \square

Consider a PILP in standard form, Q , with regions $R(t)$ and $L(t)$. There exists r such that $R(t)$ bounded in magnitude by t^r for $t \gg 0$. Define PILPs Q_α , etc. as in Proposition 3.3. In Proposition 3.3, we showed that for $t \gg 0$, the map

$$\varphi_t : L(t) \rightarrow \sqcup_{\alpha \in S} L_\alpha(t)$$

given by $x_i = \sum_{j=1}^r y_{i,j} t^{j-1}$ is a bijection. This connection between $L(t)$ and simpler lattice point sets is a critical idea in this proof.

Let $M(t)$ be the vertices of the convex hull of $L(t)$ and $M_\alpha(t)$ be defined similarly.

Proposition 5.4. *Fix $t \gg 0$ so that φ_t is a bijection. The image of $M(t)$ under φ_t lies in $\sqcup_{\alpha} M_\alpha(t)$.*

Proof. Suppose that a point p in $\sqcup_\alpha L_\alpha(t)$ is a convex combination of other points in $\sqcup_\alpha L_\alpha(t)$, say $\sum_i c_i p_i$ where for all i , $c_i \geq 0$, $p \neq p_i \in \sqcup_\alpha L_\alpha(t)$, and $\sum_i c_i = 1$. It is easy to see that φ_t^{-1} preserves this convex combination:

$$\varphi_t^{-1}(p) = \sum_i c_i \varphi_t^{-1}(p_i).$$

A point in $L(t)$ is not a convex combination of other points in $L(t)$ if and only if it is in $M(t)$. By the contrapositive of the above observation, the image under φ_t of an element of $M(t)$ is a vertex of the convex hull of $\sqcup_\alpha L_\alpha(t)$, so it is in one of the sets $M_\alpha(t)$. \square

To understand $M_\alpha(t)$, we wish to apply theorem 5.1 to $R_\alpha(t)$. Each bounding hyperplane of $R_\alpha(t)$ has the form $\mathbf{a}^\top \mathbf{y} = b(t)$, where \mathbf{a} is in \mathbb{Z}^{rn} and b has degree at most 1. As in Proposition 3.2, for all t , each vertex of $R_\alpha(t)$ is the intersection of rn of the bounding parametric hyperplanes (at t) which intersect at a single point. A size rn subset of the parametric hyperplanes has unique intersection if and only if their left hand sides form an invertible matrix, and this is independent of t . Their intersection is the inverse of this matrix times the vector of the right hand side. Since the matrix has integer entries, the intersection has the form $\mathbf{v}_{1,\mathbf{A}}t + \mathbf{v}_{2,\mathbf{A}}$ where $\mathbf{v}_{1,\mathbf{A}}, \mathbf{v}_{2,\mathbf{A}}$ are in \mathbb{Q}^{rn} i.e. is of size $O(t)$.

Proposition 5.5. *Fix t . A candidate intersection $\mathbf{v}_{1,\mathbf{A}}t + \mathbf{v}_{2,\mathbf{A}}$ is a vertex of $R_\alpha(t)$ if and only if it satisfies all of the parametric inequalities at t .*

Proof. If a candidate intersection $\mathbf{v}_{1,\mathbf{A}}t + \mathbf{v}_{2,\mathbf{A}}$ is a vertex of $R_\alpha(t)$, it lies in $R_\alpha(t)$, so it satisfies all of the parametric inequalities.

Conversely, suppose that a candidate intersection is not a vertex of $R_\alpha(t)$. One case is that it lies in $R_\alpha(t)$. We use a standard characterization of vertices of a polytope: there exists a vector \mathbf{r} such that $\mathbf{v}_{1,\mathbf{A}}t + \mathbf{v}_{2,\mathbf{A}} + s\mathbf{r}$ lies in $R_\alpha(t)$ for all reals s in some neighborhood of 0.

By construction, $\mathbf{v}_{1,\mathbf{A}}t + \mathbf{v}_{2,\mathbf{A}}$ lies on rn hyperplanes which meet at exactly one point (and possibly other hyperplanes). Therefore, at least one of these hyperplanes is not fixed by addition by \mathbf{r} . By definition, $R_\alpha(t)$ excludes all of the points on one half space determined by this hyperplane, which contradicts the previous paragraph. Therefore, the candidate does not lie in $R_\alpha(t)$, and the candidate does not satisfy all of the parametric inequalities, as desired. \square

Proposition 5.6. *Theorem 5.1 applies to $R_\alpha(t)$.*

Proof. Proposition 5.5 gives us a description of the vertices of $R_\alpha(t)$. A parametric inequality is the comparison of two (linear) polynomials, and the comparison of two polynomials stabilizes for $t \gg 0$. Since there are finitely many parametric inequalities and candidate intersections, for $t \gg 0$, all of the comparisons stabilise, and each candidate is either eventually a vertex of $R_\alpha(t)$ or eventually not a vertex.

For all t , $R_\alpha(t)$ is convex, so for $t \gg 0$, $R_\alpha(t)$ is the convex hull of the candidates which are eventually vertices. Each candidate is of size $O(t)$, so Theorem 5.1 applies. \square

Proposition 5.7. *Fix $t \gg 0$ so that φ_t is a bijection. Let*

$$T(t) := \bigcup_{\alpha \in S} \varphi_t^{-1}(M_\alpha(t)),$$

the union of the preimages of $M_\alpha(t)$. For $t \gg 0$, the vertices of the convex hull of $T(t)$ is exactly $M(t)$.

Proof. By Proposition 5.4, the image of $M(t)$ under φ_t lies in $\sqcup_\alpha M_\alpha(t)$, so each element of $M(t)$ is in $T(t)$. Each element of $T(t)$ is an element of $L(t)$ because its image under φ_t lies in one of the sets $L_\alpha(t)$. Each element of $M(t)$ is a vertex of the convex hull of $L(t)$ and lies in $T(t)$, so it is a vertex of the convex hull of $T(t)$.

Assume, for the sake of contradiction, that some vector \mathbf{v} is a vertex of the convex hull of $T(t)$ but not in $M(t)$. Then \mathbf{v} is in $L(t)$ but not in $M(t)$, so it is a convex combination of the elements of $M(t)$. $M(t)$ is a subset of $T(t)$, so \mathbf{v} in $T(t)$ is a convex combination of other elements of $T(t)$, a contradiction. Therefore, for $t \gg 0$, the vertices of the convex hull of $T(t)$ is exactly $M(t)$. \square

By Proposition 5.6, for each α , there is a positive integer d such that when t is restricted to a residue class $(\text{mod } d)$, the number of vertices of the convex hull of $L_\alpha(t)$ is eventually constant, and they are eventually given by polynomial vectors. Since there are finitely many α , we can find d that satisfies this property for all α .

Restrict t to a single residue class $(\text{mod } d)$. By Theorem 5.1, for $t \gg 0$, we can write $M_\alpha(t)$ as $\{\mathbf{y}_{\alpha,\beta}(t) \mid \beta \in S_\alpha\}$ where S_α is finite and $\mathbf{y}_{\alpha,\beta}$ are distinct vectors in $\mathbb{R}[u]^n$. Then for $t \gg 0$ (and t restricted to the residue class), $T(t)$ is a finite set of polynomial vectors.

By Proposition 5.7, for $t \gg 0$, the convex hull of $T(t)$ is exactly $M(t)$. Theorem 5.2 can be proven with the following proposition.

Proposition 5.8. *Let $\mathbf{w}_1, \dots, \mathbf{w}_p$ be distinct elements of $\mathbb{R}[u]^n$. There exists a subset U of $\{1, \dots, p\}$ such that for $t \gg 0$ $\{\mathbf{w}_h(t) \mid h \in U\}$ is the set of vertices of the convex hull of $\mathbf{w}_1(t), \dots, \mathbf{w}_p(t)$ with no repeats.*

Corollary 5.8.1. *Theorem 5.2*

Proof. We have reduced Theorem 5.2 to showing that the vertices of the convex hull of the lattice point set of a PILP in standard form has EQP structure. For $t \gg 0$, $M(t)$ equals the convex hull of $T(t)$. For t restricted to a residue class $(\text{mod } d)$ and $t \gg 0$, there is a finite set of polynomial vectors that gives $T(t)$. We can eliminate the identical polynomial vectors. By Proposition 5.8, for $t \gg 0$ (and t restricted to this residue class), the set of vertices of the convex hull of $T(t)$ is a fixed subset of polynomial vectors. This is true for all single residue classes $(\text{mod } d)$, so $M(t)$ has the desired EQP structure $(\text{mod } d)$. \square

We need a few facts to prove Proposition 5.8.

Theorem 5.9 (Carathéodory's Theorem[1]). *A point which lies in the convex hull of a subset P of \mathbb{R}^d lies in a simplex with vertices in P .*

Proposition 5.10. *Let V be any subset of $\{1, \dots, p\}$. Suppose that for some positive integer t_0 , the set $\{\mathbf{w}_h(t_0) \mid h \in V\}$ is affinely independent. Then $\{\mathbf{w}_h(t) \mid h \in V\}$ is affinely independent for $t \gg 0$.*

Proof. Let X be the matrix of polynomials with columns which are $\mathbf{w}_h - \mathbf{w}_{\min(V)}$ for all h in $V \setminus \{\min(V)\}$. For all t , $\{\mathbf{w}_h(t) \mid h \in V\}$ is affinely dependent if and only if the determinants of all $|V| - 1 \times |V| - 1$ minors of X evaluated at t equal 0. There exists such a minor of X whose determinant is nonzero at t_0 . The determinant is a polynomial of t , so it is nonzero at t for $t \gg 0$, which gives the desired result \square

Proposition 5.11. *Let V be a subset of $\{1, \dots, p\}$. Suppose that the set $\{\mathbf{w}_h(t) \mid h \in V\}$ is affinely independent for $t \gg 0$. Let \mathbf{w} be in $\mathbb{R}[u]^n$. Then either $\mathbf{w}(t)$ is a convex combination of $\{\mathbf{w}_h(t) \mid h \in V\}$ for $t \gg 0$ or $\mathbf{w}(t)$ is not a convex combination of $\{\mathbf{w}_h(t) \mid h \in V\}$ for $t \gg 0$.*

This is still true if the vectors are not affinely independent for $t \gg 0$, but it is harder to prove.

Proof. By Proposition 5.10, either $\{\mathbf{w}_h(t) \mid h \in V\} \cup \{\mathbf{w}(t)\}$ is affinely independent for $t \gg 0$ or it is affinely dependent for $t \gg 0$. In the former case, \mathbf{w} is not a convex combination of $\{\mathbf{w}_h(t) \mid h \in V\}$ for $t \gg 0$, as desired.

Now consider the latter case. The set $\{\mathbf{w}_h(t) \mid h \in V\}$ is affinely independent, and $\{\mathbf{w}_h(t) \mid h \in V\} \cup \{\mathbf{w}(t)\}$ is affinely dependent for $t \gg 0$. From standard linear algebra, there exist functions $c_h(t)$ for h in V such that for $t \gg 0$, $\sum_{h \in V} c_h(t) = 1$ and

$$(4) \quad \mathbf{w}(t) = \sum_{h \in V} c_h(t) \mathbf{w}_h(t);$$

This expresses $\mathbf{w}(t)$ as an affine combination of $\{\mathbf{w}_h(t) \mid h \in V\}$. For $t \gg 0$, $c_h(t)$ is uniquely determined. Define X as in Proposition 5.10. There is a $|V| - 1 \times |V| - 1$ minor of X , call it X^* , whose determinant is a nonzero polynomial. Let P be the projection onto the corresponding $|V| - 1$ rows (the corresponding $|V| - 1$ dimensional subspace of \mathbb{R}^d). Then $\{P(\mathbf{w}_h(t)) \mid h \in V\}$ is affinely independent for $t \gg 0$. $P(\mathbf{w}(t))$ is an affine combination of $\{P(\mathbf{w}_h(t)) \mid h \in V\}$. As before, there exist functions $c_h^*(t)$ for h in V such that $\sum_{h \in V} c_h^*(t) = 1$ and

$$P(\mathbf{w}(t)) = \sum_{h \in V} c_h^*(t) P(\mathbf{w}_h(t)),$$

which are uniquely defined for $t \gg 0$. An equivalent equation is

$$P(\mathbf{w}(t) - \mathbf{w}_{\min(V)}(t)) = \sum_{h \in V \setminus \min(V)} c_h^*(t) P(\mathbf{w}_h - \mathbf{w}_{\min(V)}(t)).$$

The functions $c_h^*(t)$ for $h \in V \setminus \min(V)$ is given by multiplying the inverse the matrix associated with the right hand side by the vector of the left hand side. Specifically the vector with coordinates $c_h^*(t)$ for $h \in V \setminus \min(V)$ is given by multiplying the inverse of X^* by the left hand side. Therefore, c_h^* for $h \in V \setminus \min(V)$ is a rational function (agrees with a rational function) for $t \gg 0$. Since $\sum_{h \in V} c_h^*(t) = 1$, $C_{\min(V)}^*$ is also a rational function. On the other hand, applying the projection to (4) gives

$$P(\mathbf{w}(t)) = \sum_{h \in V} c_h(t) P(\mathbf{w}_h(t)),$$

so for h in V and $t \gg 0$, $c_h(t) = c_h^*(t)$. Therefore, the functions c_h eventually agree with rational functions.

For $t \gg 0$, $\mathbf{w}(t)$ is a convex combination of $\{\mathbf{w}_h(t) \mid h \in V\}$ if and only if for all h in V , $0 \leq c_h(t) \leq 1$. A rational function either eventually lies in $[0, 1]$ or eventually does not lie in $[0, 1]$. Therefore, either $\mathbf{w}(t)$ is a convex combination of $\{\mathbf{w}_h(t) \mid h \in V\}$ for $t \gg 0$ or $\mathbf{w}(t)$ is not a convex combination of $\{\mathbf{w}_h(t) \mid h \in V\}$ for $t \gg 0$, as desired. \square

Proof of Proposition 5.8. Let i be in $\{1, \dots, p\}$. For all t , $\mathbf{w}_i(t)$ is a vertex of the convex hull of $\mathbf{w}_1(t), \dots, \mathbf{w}_p(t)$ if and only if it is not a convex combination of $\{\mathbf{w}_h(t)\}_{h \neq i}$. By Carathéodory's Theorem, $\mathbf{w}_i(t)$ is a convex combination of $\{\mathbf{w}_h(t)\}_{h \neq i}$ if and only if it lies in a simplex with vertices in this set. By Proposition 5.10, each subset of $\{\mathbf{w}_h\}_{h \neq i}$ is either a simplex for $t \gg 0$ or not a simplex for $t \gg 0$. There are finitely many subsets, so for $t \gg 0$, these properties stabilize. By Proposition 5.11, for each subset of $\{\mathbf{w}_h\}_{h \neq i}$ which is a simplex for $t \gg 0$, $\mathbf{w}_i(t)$ is either a convex combination for $t \gg 0$ or not a convex combination for $t \gg 0$. Combining these facts tells us that either $\mathbf{w}_i(t)$ is a vertex of the convex hull of $\mathbf{w}_1(t), \dots, \mathbf{w}_p(t)$ for $t \gg 0$ or not a vertex of the convex hull for $t \gg 0$.

Let U be the set of all i such that $\mathbf{w}_i(t)$ is a vertex of the convex hull of $\mathbf{w}_1(t), \dots, \mathbf{w}_p(t)$ for $t \gg 0$. Since $\mathbf{w}_1, \dots, \mathbf{w}_p$ are distinct polynomial vectors, for $t \gg 0$, $\{\mathbf{w}_h(t) \mid h \in U\}$ is the set of vertices of the convex hull of $\mathbf{w}_1(t), \dots, \mathbf{w}_p(t)$ with no repeats, as desired. \square

Remark. Theorem 5.1 may be used to prove Theorem 3.1 using induction on ℓ . Consider a PILP in reduced canonical form. The vertices of $R(t)$ are of size $O(t)$, so Theorem 5.1 applies. Let d be prescribed by the theorem. For $\ell = 1$ and t restricted to a residue class $(\text{mod } d)$, $f_\ell(t)$ is

the maximum over the vertices of the convex hull of the lattice point set, which is a finite set of polynomial vectors, of $\mathbf{c}^\top(t)$, which is then a maximum of polynomials. One of these polynomials is maximal for $t \gg 0$; it corresponds a single polynomial vector \mathbf{v} at which the maximum is obtained for $t \gg 0$. (There may exist others.) f_ℓ is EQP when restricted to each residue class $(\bmod d)$, so f_ℓ is EQP.

Suppose that Theorem 3.1 is true for $\ell = \ell_0$ and consider $\ell = \ell_0 + 1$. Let Q be a PILP in reduced canonical form, d as in Theorem 5.1, and t restricted to a residue class $(\bmod d)$ (throughout this paragraph). The maximum value of $\mathbf{c}^\top(t)$ is obtained at some polynomial vector \mathbf{v} in $L(t)$. Suppose that there is another PILP Q' for which $L'(t) = L(t) \setminus \{\mathbf{v}(t)\}$ (for t restricted to the residue class $(\bmod d)$). Then $f_\ell(t) = f'_{\ell-1}(t)$. Intuitively, it is possible to construct Q' by adding one more constraint to Q to exclude $\mathbf{v}(t)$ because it is a vertex of the convex hull of $L(t)$ (for t restricted to the residue class $(\bmod d)$). We do not do so explicitly here. By assumption, $f'_{\ell-1}$ is EQP, so f_ℓ is EQP when restricted to each residue class $(\bmod d)$, so we are done.

Remark. The above idea easily proves that the ℓ^{th} largest without multiplicity is eventually quasi-polynomial. Again, we use induction on ℓ . $\ell = 1$ is as before. The inductive step now only involves adding the constraint $\mathbf{c}^\top(t)\mathbf{x} \leq \mathbf{c}^\top(t)\mathbf{v}(t) - 1$.

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